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BOUNDEDNESS OF FRACTIONAL OPERATORS ON L^p SPACES WITH DIFFERENT WEIGHTS

BY

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ABSTRACT. Let T_α be either the fractional integral operator $\int f(y)|x-y|^{\alpha-n}\,dy$, or the fractional maximal operator $\sup\{r^{\alpha-n}\int_{|x-y|<r}|f(y)|\,dy: r>0\}$. Given a weight w (resp. v), necessary and sufficient conditions are given for the existence of a nontrivial weight v (resp. w) such that $(\int|T_\alpha f|^q v\,dx)^{1/q} \leqslant (\int|f|^p w\,dx)^{1/p}$ holds. Weak type substitutes in limiting cases are considered.

1. Introduction. Let T_α denote either the fractional integral operator

$$I_\alpha f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n}\,dy, \qquad 0<\alpha<n,$$

or the fractional maximal operator

$$M_\alpha f(x) = \sup_{r>0} r^{\alpha-n}\int_{|x-y|<r}|f(y)|\,dy, \qquad 0 \leqslant \alpha < n.$$

For the case of $\alpha = 0$, $M_0 f$ is the well-known Hardy-Littlewood maximal function. Sometimes we write Mf rather than $M_0 f$.

In this paper we give necessary and sufficient conditions on a weight w (resp. v) for the existence of a nontrivial weight v (resp. w) such that

$$\left(\int_{\mathbf{R}^n}|T_\alpha f(x)|^q v(x)\,dx\right)^{1/q} \leqslant \left(\int_{\mathbf{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p} \tag{1.1}$$

holds for suitable values of α, p and q.

In §2 we state and prove these characterizations for the case $T_\alpha = M_\alpha$ by a constructive method. The results obtained for M_α are used in §3 in order to prove the characterizations corresponding to the fractional integral operator I_α.

For a given α, the pairs (p,q), for which our results are proved, satisfy the conditions: $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $1/q \geqslant 1/p - \alpha/n$ with the exception of the pair $(1, n/(n-\alpha))$. For this exceptional pair we obtain analogous characterizations replacing (1.1) by the weak type inequality

$$\int_{E_\lambda} v(x)\,dx \leqslant \left(\frac{1}{\lambda}\int_{\mathbf{R}^n}|f(x)|w(x)\,dx\right)^{n/(n-\alpha)},$$

where $E_\lambda = \{x: |T_\alpha f(x)| > \lambda\}$.

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For $1 \leqslant p < \infty$, $1 \leqslant q < \infty$ and $1/q < 1/p - \alpha/n$, we show in §4 that if $v > 0$ or $w < \infty$ almost everywhere, then (1.1) holds only if the other weight is trivial, i.e., $w = \infty$ or $v = 0$ a.e. on $\mathbf{R}^n$. The analysis of the case $q = \infty$ will appear elsewhere.

As one may expect, the crucial step is to prove the results for the pairs (p, q) satisfying $1/q = 1/p - \alpha/n$. This is done in Theorems 1 and 3.

J. L. Rubio de Francia [**8**] considered inequality (1.1) for $T_\alpha = I_\alpha$ and $p = q$, i.e.

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)\, dx \leqslant \int |f(x)|^p w(x)\, dx.$$

By using vector-valued inequalities he obtained characterizations for $1 < p < \infty$ and partial results for $p = 1$. The case $p = 1$ was completed in [**2**]. E. T. Sawyer [**9**] also considered this type of inequality, but his range for p and q is more restricted than ours. Results in this direction for the Hardy-Littlewood maximal function can be found in [**1**, **4** and **7**].

Now we introduce some basic notation used in the sequel. As usual, $\mathbf{R}^n$ is the n-dimensional Euclidean space. The open ball centered at x with radius r will be denoted by $B(x, r)$, and sometimes we write B_1 instead of $B(0, 1)$. Given a set A, χ_A will stand for the characteristic function of A. The conjugate exponent p' of p is the number satisfying $1/p + 1/p' = 1$. Finally, given a weight v and a Lebesgue measurable set A, we write $v(A)$ to denote $\int_A v(x)\, dx$.

2. The maximal operator M_α. We begin by giving a characterization of the weights $w(x)$ for which there exists a nontrivial weight $v(x)$ satisfying

$$\left[\int [M_\alpha f]^q v\right]^{1/q} \leqslant C\left[\int |f|^p w\right]^{1/p}$$

for suitable p and q.

For this purpose we state and prove some auxiliary results.

LEMMA 1. *Let g be a nonnegative and locally integrable function, and α such that $0 \leqslant \alpha < n$. Then the operator M_α is of weak type $(1, n/(n-\alpha))$ with weights $[Mg]^{1-\alpha/n}$ and g, respectively, that is to say,*

$$g(\{x \colon M_\alpha f(x) > \lambda\}) \leqslant C\left(\lambda^{-1}\int |f|[Mg]^{1-\alpha/n}\right)^{n/(n-\alpha)}.$$

PROOF. For $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\lambda > 0$ we define $E_\lambda = \{x \in \mathbf{R}^n \colon M_\alpha f(x) > \lambda\}$. By using a Besicovitch type covering lemma it is possible to find a sequence of balls $B_i = B(x_i, r_i)$ such that

$$r_i^{\alpha-n}\int_{B_i} |f| > \lambda \quad \text{and} \quad \chi_{E_\lambda} \leqslant \sum \chi_{B_i} \leqslant C,$$

where C is a constant depending only on the dimension n. Then we have

$$\int_{E_\lambda} g \leqslant \sum_i \int_{B_i} g \leqslant \lambda^{n/(\alpha-n)} \sum_i r_i^{-n}\left(\int_{B_i} g\right)\left(\int_{B_i} |f|\right)^{n/(n-\alpha)}.$$

Since for $x \in B_i$ the average $r_i^{-n}\int_{B_i} g$ is dominated by $CMg(x)$, it follows that

$$\int_{E_\lambda} g \leqslant \lambda^{n/(\alpha-n)} \sum_i \left(\int_{B_i} |f| [Mg]^{1-\alpha/n} \right)^{n/(n-\alpha)}$$
$$\leqslant C \left(\lambda^{-1} \int |f| [Mg]^{1-\alpha/n} \right)^{n/(n-\alpha)},$$

as we wanted to prove. □

LEMMA 2. *Let* $1 < p < \infty$ *and let* w *be a nonnegative and locally integrable function such that* $M(w^{-p'/p}) < \infty$ *a.e. and* $\int_{B_1} w^{-p'/p} > 0$. *Under these conditions the weight* $u = [M(w^{-p'/p})]^{-\beta}\chi_{B_1}$ *with* $\beta > p - 1$ *satisfies*

$$\left(\int [M_\alpha f]^q u^{q/p} \right)^{1/q} \leqslant C_{\alpha,p,q} \left(\int |f|^p w \right)^{1/p}$$

for every α *and* q *such that* $0 \leqslant \alpha < n/p$ *and* $1/q \geqslant 1/p - \alpha/n$.

PROOF. Observe that if $1/r > 1/p - \alpha/n = 1/q$, using Hölder's inequality we obtain

$$\int [M_\alpha f]^r u^{r/p} \leqslant C \left(\int [M_\alpha f]^q u^{q/p} \right)^{r/q}.$$

Then we need only consider the case $1/q = 1/p - \alpha/n$. The argument is similar to that given in [**1**] for the Hardy-Littlewood maximal function.

Let $E_k = \{x \in \mathbf{R}^n : M(w^{-p'/p})(x) \leqslant 2^k\}$. We define

$$T_k f = M_\alpha(f w^{-p'/p}) \cdot \chi_{E_k}.$$

We shall prove the inequality

$$\left(\int [T_k f]^q \right)^{1/q} \leqslant C_{\alpha,p} 2^{k/p'} \left(\int |f|^p w^{-p'/p} \right)^{1/p}. \tag{2.1}$$

Applying Lemma 1 for $g \equiv 1$ and $fw^{-p'/p}$ instead of f, we get

$$|\{T_k f > \lambda\}| \leqslant C \left(\lambda^{-1} \int f w^{-p'/p}\, dx \right)^{n/(n-\alpha)}.$$

This shows that T_k is of weak type $(1, n/(n-\alpha))$ for the measures $w^{-p'/p}\,dx$ and dx.

On the other hand, for any $r > 0$, Hölder's inequality gives

$$r^{\alpha-n} \int_{B(x,r)} |f| w^{-p'/p} \leqslant \left(r^{-n} \int_{B(x,r)} w^{-p'/p} \right)^{1-\alpha/n} \left(\int |f|^{n/\alpha} w^{-p'/p} \right)^{\alpha/n}.$$

Therefore, by definition of the set E_k, we obtain

$$\|T_k f\|_\infty \leqslant 2^{k(1-\alpha/n)} \left(\int |f|^{n/\alpha} w^{-p'/p} \right)^{\alpha/n}.$$

Applying now Marcinkiewicz' interpolation theorem (see [**10**]) we obtain (2.1).

Substituting $fw^{p'/p}$ for f in (2.1), it follows that

$$\int_{E_k}[M_\alpha f]^q \leqslant C_{\alpha,p}2^{kq/p'}\left(\int|f|^p w\right)^{q/p}. \tag{2.2}$$

The assumption that $M(w^{-p'/p}) < \infty$ a.e. implies that $|\mathbf{R}^n - \bigcup_0^\infty E_k| = 0$. Therefore,

$$\int[M_\alpha f]^q u^{q/p} = \left(\int_{E_0} + \sum_{k=1}^{\infty}\int_{E_k\setminus E_{k-1}}\right)[M_\alpha f]^q u^{q/p}\,dx.$$

Taking into account that $\int_{B_1} w^{-p'/p} > 0$, it can be easily seen that $u = [M_\alpha(w^{-p'/p})]^{-\beta}\chi_{B_1}$ is a bounded function. Then, we obtain

$$\int[M_\alpha f]^q u^{q/p} \leqslant C\sum_{k=0}^{\infty}2^{-k\beta q/p}\int_{E_k}[M_\alpha f]^q.$$

Using (2.2), this inequality implies

$$\int[M_\alpha f]^q u^{q/p} \leqslant C_{\alpha,p}\left(\int|f|^p w\right)^{q/p}\cdot\sum_{k=0}^{\infty}2^{-kq(\beta-p+1)/p}.$$

Since $\beta > p-1$, the geometric series above is convergent. This ends the proof of the lemma. □

LEMMA 3. *Let w be a nonnegative function. Assume that for $0 \leqslant \alpha < n$ and $1 < p < \infty$, the condition*

$$\int_{|x|\leqslant R} w^{-p'/p} \leqslant CR^{(n-\alpha)p'}$$

holds for every $R \geqslant 1$. Then, given a nonnegative and integrable function g supported on B_1 and $0 < q < \infty$, there exists a finite constant C such that

$$\left(\int[M_\alpha h]^q g\right)^{1/q} \leqslant C\left(\int|h|^p w\right)^{1/p}$$

holds for every function h such that $h\chi_{B(0,2)} = 0$ a.e.

PROOF. Observe that for $x \in B_1$, there exists $R \geqslant 1$ such that

$$M_\alpha h(x) \leqslant CR^{\alpha-n}\int_{B(0,R)}|h| \leqslant CR^{\alpha-n}\left(\int|h|^p w\right)^{1/p}\left(\int_{B(0,R)}w^{-p'/p}\right)^{1/p'}.$$

Then, using the hypotheses on w and g, we have

$$\left(\int[M_\alpha h]^q g\right)^{1/q} \leqslant C\left(\int|h|^p w\right)^{1/p}\cdot\left(\int g\right)^{1/q} \leqslant C\left(\int|h|^p w\right)^{1/p}. \quad □$$

LEMMA 4. *Let μ, ν be two measures defined on $\mathbf{R}^n$. Let T be a sublinear operator of weak type (r, s), $1 \leqslant r, s \leqslant \infty$, with measures $d\mu$ and $d\nu$. Then given t, $0 < t < s$, there exists a finite constant C such that for any subset $A \subset \mathbf{R}^n$ with $\nu(A) < \infty$, we have*

$$\int_A|Tf|^t\,d\nu \leqslant C\nu(A)^{1-t/s}\left(\int|f|^r\,d\mu\right)^{t/r},$$

where C depends only on t, s and the weak type constant of T.

For a proof of this Kolmogorov type inequality see [**3** and **5**]. □

Now we are ready to prove the announced characterization.

THEOREM 1. *Let w be a nonnegative function which is finite on a set of positive measure. Let $1 < p < \infty$, $0 \leqslant \alpha < n/p$ and $1/q = 1/p - \alpha/n$. Then the following conditions are equivalent*:

(a) *There exist a nonnegative function v which is positive on a set of positive measure, and a finite constant $C = C_{\alpha,p}$ such that*

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}. \tag{2.3}$$

(b) *There exists a finite constant C such that for every $R \geqslant 1$,*

$$\int_{|x| \leqslant R} w^{-p'/p} \leqslant CR^{(n-\alpha)p'}$$

holds.

PROOF. (a) *implies* (b). Without loss of generality we may assume v is integrable and supported on the unit ball. Let $w_\varepsilon = w + \varepsilon$ and $f = w_\varepsilon^{-p'/p} \cdot \chi_{B(0,R)}$. Given $z \in B(0,1)$ and $R \geqslant 1$, we have the estimate

$$M_\alpha f(z) \geqslant CR^{\alpha-n} \int_{B(0,R)} w_\varepsilon^{-p'/p}.$$

Therefore, (a) implies

$$R^{\alpha-n}\left(\int_{B(0,R)} w_\varepsilon^{-p'/p}\right)\left(\int_{B(0,1)} v\right)^{1/q} \leqslant C\left(\int_{B(0,R)} w_\varepsilon^{-p'/p}\right)^{1/p}.$$

Since the last integral is finite we obtain

$$\int_{B(0,R)} w_\varepsilon^{-p'/p} \leqslant CR^{(n-\alpha)p'}.$$

Letting ε go to zero, by Fatou's lemma, we get (b).

(b) *implies* (a). Without loss of generality we may assume $w^{-p'/p}$ is not equal to zero a.e. on B_1. Given a function f, write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$.

To deal with f_1, consider the function $w(x)(1 + |x|)^{2(n-\alpha)p}$. Simple computations show that this weight is integrable and hence it satisfies the hypotheses of Lemma 2. Therefore, if v is defined as

$$v^{p/q} = \left[M\left(w^{-p'/p}(1 + |x|)^{2(\alpha-n)p'}\right)\right]^{-\beta} \cdot \chi_{B_1},$$

by Lemma 2 we get

$$\left(\int [M_\alpha f_1]^q v\right)^{1/q} \leqslant C\left(\int |f_1|^p w \cdot (1 + |x|)^{2(n-\alpha)p}\right)^{1/p} \leqslant C\left(\int |f|^p w\right)^{1/p}. \tag{2.4}$$

Let us consider f_2. Applying Lemma 3 with $h = f_2$ and $g = v$ we get

$$\left(\int [M_\alpha f_2]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain (a). □

REMARKS. 1. Observe that in proving (a) implies (b) we did not use the condition $1/q = 1/p - \alpha/n$. Therefore if (a) holds for any $0 \leqslant \alpha < n$, $1 < p < \infty$ and $0 < q \leqslant \infty$, then (b) holds for the same values of α and p. □

2. Observe that, if, in Lemma 2, instead of the function u, we consider $u_R = [Mw^{-p'/p}]^{-\beta}\chi_{B(0,R)}$, the conclusion remains valid with a constant depending on R. In the case $1/q = 1/p - \alpha/n$ the constant behaves like $R^{(n-\alpha)\beta/p}$. Following the steps of the proof of (b) implies (a) in the last theorem with u_R instead of u, we get inequality (2.3) with weight $v_R = u_R^{q/r}$ and constant CR^s for some $s > 0$. Clearly v_R is supported on $B(0, R)$ and it is positive there. These observations allow us to construct a weight positive everywhere and satisfying (2.3) by taking

$$v = \left[M\left(w^{-p'/p}(1+|x|)^{2(\alpha-n)p'}\right)\right]^{-\beta}(1+|x|)^{-N}$$

for $\beta > p - 1$ and N large enough. Moreover, we can get an estimate on the size of v, since for $\delta > 0$ and small enough, it satisfies

$$\int v^{-\delta}(x)(1+|x|)^{-n-1}\,dx < \infty. \quad \square$$

The preceding results can be extended to other values of α, p and q. To clarify the statements we introduce the following notation. To any given pair (α, p), such that $0 \leqslant \alpha < n$ and $1 \leqslant p < \infty$, we associate a real interval as follows:

$$Q(\alpha, p) = \begin{cases} \{q\colon 1 \leqslant q < n/(n-\alpha)\} & \text{if } p = 1, \\ \{q\colon 1 \leqslant q \leqslant np/(n-\alpha p)\} & \text{if } 1 < p < n/\alpha, \\ \{q\colon 1 \leqslant q < \infty\} & \text{if } n/\alpha \leqslant p < \infty. \end{cases}$$

With this notation we have

THEOREM 2. *Let* $1 \leqslant p < \infty$, $0 \leqslant \alpha < n$, $q \in Q(\alpha, p)$ *and* w *a nonnegative function, finite on a set of positive measure. Then the following conditions are equivalent*:

(a) *There exist a nonnegative function* v *which is positive on a set of positive measure, and a finite constant* C *such that*

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

(b) *There exists a constant* $C > 0$ *such that*:

(i) $w(x) \geqslant C(1+|x|)^{\alpha-n}$ *if* $p = 1$,

(ii) *for any* $R \geqslant 1$, $\int_{|x|\leqslant R} w^{-p'/p} \leqslant CR^{(n-\alpha)p'}$ *if* $p > 1$.

Moreover, in the case $p = 1$ *and* $q = n/(n-\alpha)$, (i) *is equivalent to the existence of a* v *satisfying the corresponding weak type inequality, i.e.*

$$v(\{M_\alpha f > \lambda\}) \leqslant C\left(\lambda^{-1}\int |f| w\right)^{n/(n-\alpha)}. \tag{2.6}$$

PROOF. (a) *implies* (b). If $p > 1$ the implication follows from Remark 1 after Theorem 1.

Let $p = 1$. Without loss of generality we may assume $\int_{B_1} v = 1$. Clearly for $x \in B(0, R)$, $M_\alpha f(x) \geqslant CR^{\alpha-n}\int_{B(0,R)}|f|$. Hence, by using (a) we obtain for $R \geqslant 1$,

$$\int |f|\chi_{B(0,R)} \leqslant CR^{n-\alpha}\int |f| w.$$

Thus, if g is any function in $L^1(\mathbf{R}^n)$ and $f = gw^{-1}$, we get

$$\left|\int gw^{-1}\chi_{B(0,R)}\right| \leqslant \int |g| w^{-1}\chi_{B(0,R)} \leqslant CR^{n-\alpha}\int |g|,$$

which implies

$$\|w^{-1}\chi_{B(0,R)}\|_\infty \leqslant CR^{n-\alpha} \quad \text{for } R \geqslant 1.$$

This clearly implies (i).

(b) *implies* (a). Consider first the case $1 < p < n/\alpha$. If $q_0 = np/(n - \alpha p)$ the result is contained in Theorem 1. Observing that the weight v constructed there is a bounded function with compact support, we have, for any $q \leqslant q_0$,

$$\int [M_\alpha f]^q v^{q/q_0} \leqslant C\left(\int [M_\alpha f]^{q_0} v\right)^{q/q_0},$$

which implies the assertion.

Suppose now that $n/\alpha \leqslant p < \infty$. As in Theorem 1 we write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$.

To deal with f_1 we first observe that there always exists γ, $0 < \gamma < n$, such that $1/q \geqslant 1/p - \gamma/n > 0$. This follows by taking γ close to, but smaller than, n/p. For this γ we have $1 < p < n/\gamma$ and $q \in Q(\gamma, p)$. It turns out that $\gamma < \alpha$. Therefore, w satisfies (ii) for γ and p. Then we may apply the previous case, obtaining an integrable weight v supported in B_1 and satisfying

$$\left(\int [M_\gamma f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

Then the desired estimate for $M_\alpha f_1$ follows from the inequality

$$\chi_{B_1} M_\alpha f_1 \leqslant CM_\gamma f_1.$$

The estimate for f_2 follows from Lemma 3 for $h = f_2$ and $g = v$.

Finally, we consider the case $p = 1$. We shall first prove the weak type inequality (2.6) from which we will derive (a) using the Kolmogorov inequality (Lemma 4).

Clearly we need only prove the assertion for $w(x) = (1 + |x|)^{\alpha - n}$. For this we just take $v(x) = (1 + |x|)^{-(n+1)}$, which is a nonincreasing and integrable function. Therefore, $Mv(x) \simeq (1 + |x|)^{-n}$. Using Lemma 1 we have

$$v(\{x\colon M_\alpha f(x) > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |f|(Mv)^{1-\alpha/n}\right)^{n/(n-\alpha)} \leqslant \left(\frac{C}{\lambda}\int |f| w\right)^{n/(n-\alpha)}.$$

Let $q < n/(n - \alpha)$. Applying Lemma 4 with measures $d\mu = w\,dx$, $d\nu = v\,dx$ and $A = \mathbf{R}^n$ we obtain

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int v\right)^{1/q-(n-\alpha)/n} \int |f| w.$$

This last argument also shows that the existence of a v satisfying the weak type inequality (2.6) implies (a) for $p = 1$ and $q < n/(n - \alpha)$, which in turn implies (i) as we have seen before. □

Next, we shall characterize the weights v for which there exist a nontrivial weight w and a finite constant C satisfying

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}$$

for suitable α, p and q.

We shall introduce some auxiliary maximal functions similar to those used by Gatto and Gutierrez in [4].

Let f be a locally integrable function, $0 \leqslant \alpha < n$. We define

$$M_\alpha^1 f(x) = \sup_{0<4r<|x|+1} r^{\alpha-n}\int_{B(x,r)} |f|,$$

$$M_\alpha^2 f(x) = \sup_{\substack{0<3r<|x|+1\\ x\in B(z,r)}} r^{\alpha-n}\int_{B(z,r)} |f|,$$

$$M_\alpha^3 f(x) = \sup_{|x|+1\leqslant 4r} r^{\alpha-n}\int_{B(x,r)} |f|.$$

Since we will use M_0^2 frequently in the sequel, we write $\tilde{g}$ instead of $M_0^2(g)$. With these notations we have

LEMMA 5. *Let f, g be nonnegative and locally integrable functions, $0 \leqslant \alpha < n$, $n/(n-\alpha) < q < \infty$ and $1/p = 1/q + \alpha/n$. Then there exists a finite constant C such that*

$$\left(\int [M_\alpha^1 f]^q g\right)^{1/q} \leqslant C\left(\int f^p \tilde{g}^{p/q}\right)^{1/p}. \tag{2.7}$$

Moreover, for $q = n/(n-\alpha)$,

$$g(\{M_\alpha^1 f > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int f\tilde{g}^{1-\alpha/n}\right)^{n/(n-\alpha)} \tag{2.8}$$

holds.

PROOF. We define the operator

$$T_\alpha f = M_\alpha^1(f\tilde{g}^{\alpha/n}).$$

Let $E_\lambda = \{x\colon\ T_\alpha f(x) > \lambda\}$. By using a Besicovitch type covering lemma, it is possible to find a countable family of balls $B_i = B(x_i, r_i)$ such that

$$4r_i < |x_i| + 1, \quad \chi_{E_\lambda} \leqslant \sum_i \chi_{B_i} \leqslant C_n \quad \text{and} \quad r_i^{\alpha-n}\int_{B_i} f\tilde{g}^{\alpha/n} > \lambda.$$

Therefore,

$$\int_{E_\lambda} g \leqslant \sum_i \int_{B_i} g \leqslant \sum_i \lambda^{-n/(n-\alpha)} r_i^{-n}\left(\int_{B_i} f\tilde{g}^{\alpha/n}\right)^{n/(n-\alpha)} \int_{B_i} g.$$

Since for $x \in B_i$, $\tilde{g}(x) \geqslant r_i^{-n}\int_{B_i} g$, it follows that

$$\int_{E_\lambda} g \leqslant \lambda^{-n/(n-\alpha)} \sum_i \left(\int_{B_i} f\tilde{g}\right)^{n/(n-\alpha)} \leqslant C\left(\lambda^{-1}\int f\tilde{g}\right)^{n/(n-\alpha)}.$$

This shows that T_α is of weak type $(1, n/(n-\alpha))$ with measures $\tilde{g}\,dx$ and $g\,dx$, respectively, implying, in particular, (2.8).

On the other hand, applying Hölder's inequality, we have

$$r^{\alpha-n}\int_{B(x,r)} f\tilde{g}^{\alpha/n} \leqslant C\left(\int f^{n/\alpha}\tilde{g}\right)^{\alpha/n},$$

which proves that T_α is of strong type $(n/\alpha, \infty)$ with the corresponding measures. Applying the Marcinkiewicz interpolation theorem, we obtain

$$\left(\int [T_\alpha f]^q g\right)^{1/q} \leqslant C\left(\int f^p \tilde{g}\right)^{1/p},$$

or, equivalently,

$$\left(\int [M_\alpha^1 f]^q\right)^{1/q} \leqslant C\left(\int f^p \tilde{g}^{p/q}\right)^{1/p}.$$

This proves (2.7). □

THEOREM 3. *Let v be a nonnegative function, different from zero on a set of positive measure. Let $0 \leqslant \alpha < n$, $n/(n-\alpha) < q < \infty$ and $1/p = 1/q + \alpha/n$. Then the following conditions are equivalent.*

(a) *There exist a nonnegative function w finite on a set of positive measure, and a finite constant C such that*

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

(b)

$$\int v(x)(1+|x|)^{(\alpha-n)q}\,dx < \infty.$$

PROOF. (a) *implies* (b). There always exist a ball $B = B(0, R)$ and a function f, supported on B, such that $\int |f|^p w < \infty$ and $\int_B |f| = 1$. It is easy to see that for this function f, the inequality $M_\alpha f(x) \geqslant C(1+|x|)^{\alpha-n}$ holds. Hence, condition (a) implies $\int v(x)(1+|x|)^{(\alpha-n)q}\,dx < \infty$.

(b) *implies* (a). According to the definitions of M_α^1 and M_α^3, we have $M_\alpha f \leqslant M_\alpha^1 f + M_\alpha^3 f$. Now, using Hölder's inequality, we obtain

$$M_\alpha^3 f(x) \leqslant C(1+|x|)^{\alpha-n}\left(\int |f(y)|^p(1+|y|)^{\beta p}\,dy\right)^{1/p}\left(\int (1+|y|)^{-\beta p'}\,dy\right)^{1/p'}.$$

By choosing $\beta > n/p'$, the last integral is finite. These estimates together with Lemma 5 give

$$\left(\int [M_\alpha f]^q v\right)^{p/q} \leqslant C\left(\int v(x)(1+|x|)^{(\alpha-n)q}\,dx\right)^{p/q}\int |f(y)|^p(1+|y|)^{\beta p}\,dy$$
$$+ C\int |f|^p \tilde{v}^{p/q} \leqslant C\int |f|^p w,$$

where w can be taken as $w(y) = [\tilde{v}(y) + (1 + |y|)^{\beta q}]^{p/q}$. Since v is locally integrable, w is finite almost everywhere. □

REMARKS. 1. The proofs of (a) implies (b) and of the boundedness of M_α^3 as an operator from $L^p(\mathbf{R}^n, w\,dx)$ to $L^q(\mathbf{R}^n, v\,dx)$ remain valid for any α, p and q such that $0 \leqslant \alpha < n$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. □

2. For N large enough, the weight constructed in Theorem 3 satisfies

$$\int w(x)(1 + |x|)^{-N}\,dx < \infty.$$

To prove this, it is enough to show that $\int \tilde{v}^{p/q}(x)(1 + |x|)^{-N}\,dx$ is finite. To see this let $I = [1,\ 2)$, $v_i = v\chi_I(|x|2^{-i})$ if $i = 1, 2, \ldots$; $v_0 = v\chi_{B(0,2)}$. It is easy to check that $\tilde{v}_0$ is supported on $B(0, 8)$ and $\tilde{v}_i$ is supported on the set of x such that $2^{i-2} \leqslant |x| < 2^{i+3}$. Therefore, since $p < q$, we have

$$\int \tilde{v}^{p/q}(x)(1 + |x|)^{-N}\,dx \leqslant C \sum_{i=0}^{\infty} 2^{-iN} \int_{B(0,2^{i+3})} \tilde{v}_i^{p/q}.$$

Since M_0^2 is of weak type (1,1), applying Kolmogorov's inequality (Lemma 4), the right-hand side is bounded by

$$C \sum_{i=0}^{\infty} 2^{i(n-np/q-N)} \left(\int v_i\right)^{p/q} \leqslant C\left(\int v(x)(1 + |x|)^{(\alpha-n)q}\,dx\right)^{p/q} \sum_{i=0}^{\infty} 2^{i(np-N)},$$

which is finite if $N > np$. □

As in the case of Theorem 1, the preceding result can be extended to a wider range of p and q. It will be convenient to associate to any given pair (α, q), $0 \leqslant \alpha < n$, and $1 \leqslant q < \infty$, a real interval $P(\alpha, q)$ defined by

$$P(\alpha, q) = \begin{cases} \{p\colon 1 \leqslant p < \infty\} & \text{if } 1 \leqslant q < n/(n - \alpha), \\ \{p\colon 1 < p < \infty\} & \text{if } q = n/(n - \alpha), \\ \{p\colon qn/(n + \alpha q) \leqslant p < \infty\} & \text{if } n/(n - \alpha) < q < \infty. \end{cases}$$

Using this notation we state

THEOREM 4. *Let $0 \leqslant \alpha < n$, $1 \leqslant q < \infty$, $p \in P(\alpha, q)$ and v be a nonnegative function different from zero on a set of positive measure. Then the following conditions are equivalent*:

(a) *There exist a nonnegative function w, finite on a set of positive measure, and a finite constant C such that*

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

(b)

$$\int v(x)(1 + |x|)^{(\alpha-n)q}\,dx < \infty.$$

Moreover, in the case $q = n/(n - \alpha)$, $p = 1$, the existence of w satisfying the weak type inequality

$$v(\{M_\alpha f > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |f| w\right)^{n/(\alpha-n)} \tag{2.9}$$

is equivalent to $\sup_{R \geqslant 1} R^{-n} \int_{B(0,R)} v < \infty$.

PROOF. (a) *implies* (b). This implication follows from Remark 1 after Theorem 3.

(b) *implies* (a). We first consider the case $n/(n-\alpha) < q < \infty$ and $p \in P(\alpha, q)$. By Theorem 3, if $p_0 = qn/(n+\alpha q)$, there exists a nontrivial weight w such that

$$\left(\int [M_\alpha f]^q v\right)^{p_0/q} \leqslant C\int |f|^{p_0} w.$$

Using Hölder's inequality we get

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int |f(x)|^p w(x)^{p/p_0}(1+|x|)^{(n+1)p/p_0}\, dx\right)^{1/p}.$$

This proves (b) implies (a) in this case.

Let us now consider $1 \leqslant q \leqslant n/(n-\alpha)$, $p \in P(\alpha, q)$ and $p > 1$. Then there exists r, $1 < r < \infty$, such that $1/p - \alpha/n \leqslant 1/r < 1 - \alpha/n$. By an application of Hölder's inequality we have

$$\int [M_\alpha f]^q v \leqslant \left(\int [M_\alpha f]^r (1+|x|)^{\delta r/q} v\right)^{q/r} \left(\int (1+|x|)^{-\delta r/(r-q)} v\right)^{(r-q)/r}.$$

Choosing $\delta = q(n-\alpha)(r-q)/r$, it follows from (b) that the last integral is finite and the weight $v_1(x) = (1+|x|)^{\delta r q} v(x)$ satisfies

$$\int v_1(x)(1+|x|)^{(\alpha-n)r}\, dx < \infty.$$

Therefore v_1, r, p and α satisfy (b). Since $r > n/(n-\alpha)$, applying the previous case, we can find a nontrivial weight w such that

$$\left(\int [M_\alpha f]^q v\right)^{1/q} \leqslant C\left(\int [M_\alpha f]^r v_1\right)^{1/r} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

Finally we consider $p = 1$. Let $q < n/(n-\alpha)$. As before we estimate M_α by $M_\alpha^1 + M_\alpha^3$. Clearly

$$M_\alpha^3 f(x) \leqslant C(1+|x|)^{\alpha-n}\int |f|.$$

Therefore, if v satisfies (b) we obtain

$$\int [M_\alpha^3 f]^q v \leqslant C\left(\int |f|\right)\left(\int (1+|x|)^{(\alpha-n)q} v(x)\, dx\right) \leqslant C\int |f|. \tag{2.10}$$

On the other hand, inequality (2.8) of Lemma 5 gives

$$v(\{M_\alpha^1 h > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |h| \tilde{v}^{1-\alpha/n}\right)^{n/(n-\alpha)}. \tag{2.11}$$

Let us take $A_0 = B_1$, $A_k = \{x\colon 2^{k-1} < |x| < 2^k\}$ if $k = 1, 2, \ldots$; $\chi_k = \chi_{A_k}$ and $f_k = f\chi_k$. Observe that the support of $M_\alpha^1 f_k$ is contained in a ball B_k with finite radius. Then, applying Kolmogorov's inequality to (2.11), one obtains

$$\left(\int_{B_k} [M_\alpha^1 h]^q v\right)^{1/q} \leqslant C_k \int |h| \tilde{v}^{1-\alpha/n}.$$

Therefore

$$\left(\int [M_\alpha^1 f]^q v\right)^{1/q} \leqslant \sum_k \left(\int [M_\alpha^1 f_k]^q v\right)^{1/q} \leqslant \sum_k C_k \int |f_k| \tilde{v}^{1-\alpha/n}$$
$$= \int |f| \left(\sum_k C_k \chi_k\right) \tilde{v}^{1-\alpha/n}.$$

The last estimate combined with (2.10) gives (a) for $w = 1 + \tilde{v}^{1-\alpha/n}\sum_k C_k\chi_k$. To finish the proof of the theorem suppose that the weak type inequality (2.9) holds. As before, we take f such that $M_\alpha f(x) \geqslant C(1+|x|)^{\alpha-n}$ and $\int |f| w < \infty$. Then, using (2.9), we get

$$v\left(\left\{x\colon (1+|x|)^{\alpha-n} > \lambda\right\}\right) \leqslant C\lambda^{n/(\alpha-n)}.$$

But $\{x\colon (1+|x|)^{\alpha-n} > \lambda\} = B(0, \lambda^{1/(\alpha-n)} - 1)$ for $\lambda < 1$. Therefore for $R \geqslant 1$, $\int v\chi_{B(0,R)} \leqslant CR^n$.

Conversely, if v satisfies the last property we may apply (2.8) to estimate $M_\alpha^1(f)$. Since we also have $M_\alpha^3 f \leqslant C_0(1+|x|)^{\alpha-n}\int |f|$, assuming $\int |f| = C_0^{-1}$ we obtain

$$v\left(\left\{x\colon M_\alpha^3 f(x) > \lambda\right\}\right) \leqslant v\left(\left\{x\colon (1+|x|)^{\alpha-n} > \lambda\right\}\right).$$

Now, if $\lambda \geqslant 1$ the last set is empty, so we need only consider $\lambda < 1$. In this case the right-hand side is bounded by $v(B(0, \lambda^{1/(\alpha-n)}))$, which is less than or equal to $C\lambda^{n/(\alpha-n)}$, by assumption. Then, taking $w = 1 + \tilde{v}^{1-\alpha/n}$, we obtain the weak type inequality (2.9). This concludes the proof of the theorem. □

3. Fractional integral operators. As in the preceding section, we shall begin by studying the weights w for which there exists a nontrivial v satisfying $(\int |I_\alpha f|^q v)^{1/q} \leqslant C(\int |f|^p w)^{1/p}$ for suitable α, p and q. A characterization of this class of weights is given in the following theorem where we use the notation introduced in §2.

THEOREM 5. *Let w be a nonnegative function finite on a set of positive measure. Let $0 < \alpha < n$, $1 \leqslant p < \infty$, $q \in Q(\alpha, p)$. Then the following conditions are equivalent*:

(a) *There exist a nonnegative function v, different from zero on a set of positive measure, and a finite constant C such that*

$$\left(\int |I_\alpha f|^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

(b) (i) $\int w(x)^{-p'/p}(1+|x|)^{(\alpha-n)p'}\,dx < \infty$ *if* $p > 1$,

(ii) $w(x) \geqslant C(1+|x|)^{\alpha-n}$ *if* $p = 1$.

Moreover, the existence of v satisfying the weak type inequality

$$v(\{|I_\alpha f| > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |f| w\right)^{n/(n-\alpha)} \tag{3.1}$$

is equivalent to (ii).

PROOF. (a) *implies* (b). Without loss of generality we may assume $\int_{B_1} v = 1$. For any locally integrable function f, if $|x| < 1$, we have

$$(I_\alpha |f|)(x) \geqslant \int |f(y)|(1+|y|)^{\alpha-n}\,dy.$$

Then, (a) gives

$$\int |f(y)|(1+|y|)^{\alpha-n}\,dy \leqslant C\left(\int |f|^p w\right)^{1/p}$$

or, equivalently,

$$\int |f(y)|w(y)^{-1/p}(1+|y|)^{\alpha-n}\,dy \leqslant C\left(\int |f|^p\right)^{1/p}.$$

Therefore we obtain

$$\int w(y)^{-p'/p}(1+|y|)^{(\alpha-n)p'}\,dy < \infty \quad \text{for } p > 1$$

and

$$w(x) \geqslant C(1+|x|)^{\alpha-n} \quad \text{a.e. for } p = 1.$$

(b) *implies* (a). Without loss of generality we may assume $w^{-1}\chi_{B_1}$ is positive on a set of positive measure. Let us decompose $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$.

First we estimate $I_\alpha f_2$. It is clear that

$$|I_\alpha f_2|\chi_{B_1} \leqslant C\int |f_2(y)|(1+|y|)^{\alpha-n}\,dy. \tag{3.2}$$

Then, for any integrable function v supported in B_1, we have

$$\begin{aligned}\left(\int |I_\alpha f_2|^q v\right)^{1/q} &\leqslant C\int |f_2(y)|(1+|y|)^{\alpha-n}\,dy\\ &\leqslant C\left(\int |f|^p w\right)^{1/p}\left(\int w^{-p'/p}(1+|y|)^{(\alpha-n)p'}\right)^{1/p'} \quad \text{if } p > 1\end{aligned}$$

or

$$\left(\int |I_\alpha f|^q v\right)^{1/q} \leqslant C\int |f_2(y)|(1+|y|)^{\alpha-n}\,dy \quad \text{if } p = 1.$$

Therefore, if w satisfies either (i) or (ii), we have

$$\left(\int |I_\alpha f_2|^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}. \tag{3.3}$$

In order to estimate $I_\alpha f_1$ we shall make use of the following inequality (see [**6** and **9**]), valid for $\varepsilon > 0$ and small enough

$$|I_\alpha f| \leqslant C[M_{\alpha-\varepsilon}f \cdot M_{\alpha+\varepsilon}f]^{1/2}, \tag{3.4}$$

where C depends upon ε and α.

We first consider $1 < p < n/\alpha$. Let us define the function W by

$$W(y)^{-p'/p} = w(y)^{-p'/p}(1+|y|)^{(\alpha-n)p'}.$$

Thus, if w satisfies (i) the weight W is such that $M(W^{-p'/p}) < \infty$ a.e. Therefore, by Lemma 2 the weight $V = [M(W^{-p'/p})]^{-\beta}\chi_{B_1}$ satisfies

$$\left(\int [M_\gamma f]^r V^{r/p}\right)^{1/r} \leqslant C_{\gamma,p,r}\left(\int |f|^p W\right)^{1/p} \tag{3.5}$$

for every γ and r such that $0 < \gamma < n/p$ and $r \in Q(\gamma, p)$.

Let $\varepsilon > 0$, $\alpha_1 = \alpha - \varepsilon$, $\alpha_2 = \alpha + \varepsilon$. If ε is small enough we get $0 < \alpha_i < n/p$. Let q_1, q_2 be such that $1/q_1 = 1/q + \varepsilon/n$, $1/q_2 = 1/q - \varepsilon/n$. It follows that $q_i \in Q(\alpha_i, p)$. Therefore, (3.5) implies

$$\left(\int [M_{\alpha_i} f]^{q_i} v_i\right)^{1/q_i} \leqslant C\left(\int |f|^p W\right)^{1/p},$$

where $v_i = V^{q_i/p}$.

Then, applying (3.4) and Hölder's inequality with exponents $2q_i/q$, we obtain

$$\int |I_\alpha f|^q v^{q/p} \leqslant C\left(\int [M_{\alpha_1} f]^{q_1} v_1\right)^{q/2q_1}\left(\int [M_{\alpha_2} f]^{q_2} v_2\right)^{q/2q_2} \leqslant C\left(\int |f|^p W\right)^{q/p}.$$

In particular, applying this inequality to f_1 we have

$$\left(\int |I_\alpha f_1|^q v\right)^{1/q} \leqslant C\left(\int |f_1|^p w\right)^{1/p} \tag{3.6}$$

with $v = V^{q/p}$.

In order to finish the estimate for $I_\alpha f_1$ we need to find a v satisfying (3.6) for the cases $p = 1$ and $n/\alpha \leqslant p < \infty$. Clearly for any $\delta \leqslant \alpha$ we have

$$|I_\alpha f_1| \chi_{B_1} \leqslant (I_\delta |f_1|) \chi_{B_1}. \tag{3.7}$$

It is easy to see that if $0 < \alpha < n$ and $q \in Q(\alpha, p)$ there exists $\varepsilon > 0$ such that $0 < \alpha - 2\varepsilon$ and $q \in Q(\alpha - 2\varepsilon, p)$. Moreover, if w satisfies (b) for (α, p), w also satisfies Theorem 2(b) for (α, p) and $(\alpha - 2\varepsilon, p)$. Therefore, applying that theorem, there exists a nontrivial weight v, integrable and supported in B_1, such that

$$\left(\int [M_\gamma h]^q v\right)^{1/q} \leqslant C\left(\int |h|^p w\right)^{1/p}, \qquad \gamma = \alpha, \alpha - 2\varepsilon.$$

Then, by using (3.7), (3.4) and the last inequality we obtain

$$\begin{aligned} \int |I_\alpha f_1|^q v &\leqslant C \int [I_{\alpha-\varepsilon} |f_1|]^q v \\ &\leqslant C\left(\int [M_{\alpha-2\varepsilon} f_1]^q v\right)^{1/2}\left(\int [M_\alpha f_1]^q v\right)^{1/2} \leqslant C\left(\int |f_1|^p w\right)^{q/p}. \end{aligned} \tag{3.8}$$

Putting together (3.3), (3.6) and (3.8) we get (b) implies (a), as stated.

It remains to consider the case $p = 1$, $q = n/(n - \alpha)$. Assume w satisfies (ii). Writing, as above, $f = f_1 + f_2$, (3.2) shows that (3.3) also holds in this case with, say, $v = \chi_{B_1}$. Thus, we need only prove (3.1) for f_1. But, since I_α is of weak type $(1, n/(n - \alpha))$ with Lebesgue measure, we have

$$|\{x: |(I_\alpha f)(x)| > \lambda\}| \leqslant \left(\frac{C}{\lambda}\int |f_1|\right)^{n/(n-\alpha)} \leqslant \left(\frac{C}{\lambda}\int |f_1|(y)|(1 + |y|)^{\alpha - n}\, dy\right)^{n/(n-\alpha)}.$$

Then (ii) implies (3.1) for $v = \chi_{B_1}$.

Finally, if there exists a nontrivial weight v satisfying (3.1), since we may always assume v is integrable, we can apply Kolmogorov's inequality (Lemma 4) to obtain

$$\left(\int |I_\alpha f|^q v\right)^{1/q} \leqslant C\int |f| w$$

for $1 \leqslant q < n/(n-\alpha)$. Since, for these values of q, (a) implies (ii), the proof of the theorem is complete. □

The next theorem gives a characterization of the weights v for which there exists a nontrivial weight w such that I_α is a bounded operator from $L^p(w\,dx)$ to $L^q(v\,dx)$.

THEOREM 6. *Let v be a nonnegative function which is different from zero on a set of positive measure. Let $0 < \alpha < n$, $1 \leqslant q < \infty$ and $p \in P(\alpha, q)$. Then the following conditions are equivalent*:

(a) *There exist a nonnegative function w which is finite on a set of positive measure, and a finite constant C such that*

$$\left(\int |I_\alpha f|^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

(b)

$$\int v(x)(1+|x|)^{(\alpha-n)q}\,dx < \infty.$$

Moreover, for $q = n/(n-\alpha)$, the existence of w satisfying

$$v(\{I_\alpha f > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |f| w\right)^{n/(n-\alpha)} \tag{3.9}$$

is equivalent to $\sup_{R\geqslant 1} R^{-n}\int_{B(0,R)} v < \infty$.

PROOF. (a) *implies* (b). There always exist a ball $B = B(0, R)$ and a nonnegative function supported on B such that $\int f^p w < \infty$ and $\int f = 1$. For this function we have

$$I_\alpha f(x) \geqslant (R+|x|)^{\alpha-n}\int_B f \geqslant C(1+|x|)^{\alpha-n}.$$

Then using (a) we obtain (b).

(b) *implies* (a). We first suppose $1 \leqslant q < n/(n-\alpha)$. Let us take $\varepsilon > 0$ and define $\alpha_1 = \alpha - \varepsilon$, $\alpha_2 = \alpha + \varepsilon$. By choosing ε small enough we have $0 < \alpha_i < n$, $1/q > 1 - \alpha_i/n$ and, hence, $P(\alpha_i, q) = P(\alpha, q)$. Moreover, the weights

$$v_1(x) = v(x)(1+|x|)^{\varepsilon q} \quad \text{and} \quad v_2(x) = v(x)(1+|x|)^{-\varepsilon q}$$

satisfy Theorem 4(b) for the pairs (α_1, q), (α_2, q), respectively. Therefore there exists a nontrivial weight w such that

$$\left(\int [M_{\alpha_i} f]^q v_i\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p}.$$

Using (3.4) we obtain

$$\int |I_\alpha f|^q v \leqslant C\left(\int [M_{\alpha_1} f]^q v_1\right)^{1/2}\left(\int [M_{\alpha_2} f]^q v_2\right)^{1/2} \leqslant C\left(\int |f|^p w\right)^{q/p},$$

which proves (a).

Suppose now $n/(n-\alpha) \leqslant q < \infty$. We take as above $\varepsilon > 0$, $\alpha_1 = \alpha - \varepsilon$, $\alpha_2 = \alpha + \varepsilon$. We also define q_1 and q_2 by $1/q_1 = 1/q + \varepsilon/n$, $1/q_2 = 1/q - \varepsilon/n$. Taking ε small enough we can make $0 < \alpha_i < n$ and $1 < q_i < \infty$. Moreover,

$$1/q = 1 - \alpha/n \quad \text{if and only if} \quad 1/q_i = 1 - \alpha_i/n,$$

$$1/q > 1 - \alpha/n \quad \text{if and only if} \quad 1/q_i > 1 - \alpha_i/n \text{ and } 1/q_i + \alpha_i/n = 1/q + \alpha/n.$$

Therefore $P(\alpha_i, q_i) = P(\alpha, q)$. Taking now $\delta_1 = \varepsilon q_1[1 - q(n-\alpha)/n]$ and $\delta_2 = -\varepsilon q_2[1 - q(n-\alpha)/n]$, we have that the weights $v_i(x) = v(x)(1+|x|)^{\delta_i}$ satisfy Theorem 4(b) for the pairs (α_i, q_i). Therefore for $p \in P(\alpha, q)$ we can find a weight w satisfying

$$\left(\int \left[M_{\alpha_i} f\right]^{q_i} v_i\right)^{1/q_i} \leqslant C\left(\int |f|^p w\right)^{1/p}, \qquad i = 1, 2. \tag{3.10}$$

Again using (3.4) and Hölder's inequality for $2q_i/q$ we obtain

$$\int |I_\alpha f|^q v \leqslant C\left(\int \left[M_{\alpha_1} f\right](x)^{q_1}(1+|x|)^{\theta 2q_1/q} v(x)\,dx\right)^{q/2q_1}$$
$$\cdot\left(\int \left[M_{\alpha_2} f\right](x)^{q_2}(1+|x|)^{-\theta 2q_2/q} v(x)\,dx\right)^{q/2q_2}.$$

Taking $\theta = \varepsilon q[n - (n-\alpha)q]/2n$, the last term is equal to

$$C\left(\int \left[M_{\alpha_1} f\right]^{q_1} v_1\right)^{q/2q_1} \cdot \left(\int \left[M_{\alpha_2} f\right]^{q_2} v_2\right)^{q/2q_2}.$$

Then, using (3.10), (a) follows.

Finally let $q = n/(n-\alpha)$ and $p = 1$. First we assume that for $R \geqslant 1$, $\int v\chi_{B(0,R)}$ is smaller than a constant times R^n and define α_i and q_i as above. Then we can apply Theorem 4 to the weight v for the pairs (α_i, q_i) to obtain a weight w and a finite constant C satisfying

$$v(\{M_{\alpha_i} f > \lambda\}) \leqslant \left(\frac{C}{\lambda}\int |f| w\right)^{q_i}. \tag{3.11}$$

Now it is not difficult to see that given any $\delta > 0$ and ε small enough we have the estimates

$$|I_\alpha^1 f|(x) = \left|\int_{|x-y|<\delta} f(y)|x-y|^{\alpha-n}\,dy\right| \leqslant C\delta^{\varepsilon}(M_{\alpha_1} f)(x),$$

$$|I_\alpha^2 f|(x) = \left|\int_{|x-y|\geqslant\delta} f(y)|x-y|^{\alpha-n}\,dy\right| \leqslant C\delta^{-\varepsilon}(M_{\alpha_2} f)(x).$$

Therefore

$$v(\{|I_\alpha f| > \lambda\}) \leqslant v(\{M_{\alpha_1} f > \lambda/2C\delta^{\varepsilon}\}) + v(\{M_{\alpha_2} f > \lambda/2C\delta^{-\varepsilon}\}).$$

Hence (3.11) implies

$$v(\{|I_\alpha f| > \lambda\}) \leqslant C\left[\left(\frac{\delta^{\varepsilon}}{\lambda}\int |f| w\right)^{q_1} + \left(\frac{\delta^{-\varepsilon}}{\lambda}\int |f| w\right)^{q_2}\right].$$

Then taking $\delta = (\lambda^{-1}\int |f| w)^{q/n}$, we obtain (3.9).

Conversely, if v and w satisfy (3.9), an argument of the same type that we used to prove the necessary condition for (2.9) in Theorem 4 shows that v satisfies $\sup_{R \geqslant 1} R^{-n} \int_{B(0,R)} v < \infty$. □

4. Comments. I. We would like to point out that the corresponding conditions (a) in Theorems 4 and 6 are equivalent to the seemingly stronger conditions:

(a′) There exist $w < \infty$ a.e. and a finite constant C such that $(\int |Tf|^q v)^{1/q} \leqslant C(\int |f|^p w)^{1/p}$ for T equals to M_α or I_α, respectively.

Clearly (a′) implies (a) in the corresponding theorems. On the other hand, if the weight v satisfies either Theorem 4(b) or Theorem 6(b), it is easy to see that the weights w constructed in proving them are finite almost everywhere, which gives (a′).

Similarly conditions (a) in Theorems 2 and 5 are equivalent to:

(a″) There exist $v > 0$ a.e. and a finite constant C such that

$$\left(\int |Tf|^q v\right)^{1/q} \leqslant C\left(\int |f|^p w\right)^{1/p} \tag{4.1}$$

for T equals M_α or I_α, respectively.

As before, it is obvious that (a″) implies (a). To prove that (a) implies (a″), since we already know that (b) is equivalent to (a), it is enough to show that (b) implies (a″). Let $\{Q_k\}$ be a partition of $\mathbf{R}^n$ into cubes. If w is almost everywhere finite on $\mathbf{R}^n$, and proceeding as in the proofs of Theorems 2 and 5, we can get for every k a weight v_k, supported and positive on Q_k, such that

$$\int |Tf|^q v_k \, dx \leqslant \left(\int |f|^p w \, dx\right)^{q/p}.$$

Multiplying these inequalities by 2^{-k} and adding them up, we get that $v = \sum 2^{-k} v_k$ is positive a.e. on $\mathbf{R}^n$ and satisfies (4.1). This proves (a″). In fact, if w is not necessarily finite almost everywhere, but satisfies condition (b) of either Theorem 2 or Theorem 5, the weight w_1 defined by $w_1^{-1}(x) = w^{-1}(x) + (1 + |x|)^{-\eta}$ also satisfies (b) for η large enough. Clearly, $w_1 < w$ and w_1 is finite almost everywhere, which reduces the problem to the previous case.

II. Now we want to discuss the range of values of p and q for which the results have been established. We already pointed out that the proofs of (b) implies (a) in the preceding theorems were valid without restrictions on p and q. In the next theorem we show that either (a′) or (a″) cannot hold for pairs (p, q), $1 \leqslant p, q < \infty$, other than the ones given in Theorems 2, 4, 5 and 6.

THEOREM 7. *Let* $1/q < 1/p - \alpha/n$. *Consider the inequality*

$$\left(\int |T_\alpha f|^q v\right)^{1/q} \leqslant \left(\int |f|^p w\right)^{1/p}, \tag{4.2}$$

where T_α *is either* I_α *or* M_α. *Then*:

(i) *There exist* $w < \infty$ *a.e. and a nonnegative* v *such that* (4.2) *holds only if* $v = 0$ *a.e.*

(ii) *There exist* $v > 0$ *a.e. and a* w *such that* (4.2) *holds only if* $w = \infty$ *a.e.*

PROOF. In order to prove the theorem it is enough to show that for any given pair of weights v and w such that the set $E = \{x\colon v(x) > 0\} \cap \{x\colon w(x) < \infty\}$ has positive measure, (4.2) is false whenever $1/q < 1/p - \alpha/n$. Under this assumption on E there exist a number N and a ball B such that the measure of the set $G = \{x\colon x \in B, v(x) > N^{-1}, w(x) < N\}$ is positive. Let x_0 be a point of density for G. Without loss of generality we may assume $x_0 = 0$ and $B = B_1$.

We shall prove the theorem for the case $T_\alpha = I_\alpha$. The proof of the case $T_\alpha = M_\alpha$ is similar.

Let β be a number satisfying $1/q < (\beta - \alpha)/n < 1/p - \alpha/n$ and let $f(y) = |y|^{-\beta}\chi_G(y)$. Let $|x| < 1$. Then,

$$I_\alpha f(x) = \int_G |y|^{-\beta}|x-y|^{\alpha-n}\,dy \geqslant \int_G |y|^{-\beta}(|x| + |y|)^{\alpha-n}\,dy$$

$$\geqslant |x|^{\alpha-\beta}\int |t|^{-\beta}(1+|t|)^{\alpha-n}\chi_G(|x|t)\,dt.$$

Since 0 is a point of density for G, given η positive and small enough, there exists $\delta > 0$ such that $r^{-n}|G \cap B_r| > \eta$ holds for every $0 < r \leqslant \delta$. Therefore,

$$\begin{aligned} I_\alpha f(x) &\geqslant |x|^{\alpha-\beta}\int_{|t|\leqslant\delta} |t|^{-\beta}(1+|t|)^{\alpha-n}\chi_G(|x|t)\,dt \\ &\geqslant C|x|^{\alpha-\beta}|\{t\colon t|x| \in G, t \in B_\delta\}| \\ &= C|x|^{\alpha-\beta}|x|^{-n}|G \cap B_{|x|\delta}| \geqslant C'\eta|x|^{\alpha-\beta}. \end{aligned}$$

By definition of the set G, we have

$$\int |I_\alpha f|^q v\,dx \geqslant N^{-1}\int_G |I_\alpha f|^q\,dx \geqslant N^{-1}(C'\eta)^q\int_G |x|^{(\alpha-\beta)q}\,dx.$$

Let $\varepsilon(r) = [|B_r| - |B_r \cap G|]^{1/n}$. The assumption that 0 is a point of density for G tells us that $\varepsilon(r)/r$ tends to zero for r tending to zero. Thus, since $|x|^{(\alpha-\beta)q}$ is a decreasing function of $|x|$, we get

$$\begin{aligned} \int_G |x|^{(\alpha-\beta)q}\,dx &\geqslant \int_{G\cap B_r} |x|^{(\alpha-\beta)q}\,dx \geqslant \int_{\varepsilon(r)<|x|\leqslant r} |x|^{(\alpha-\beta)q}\,dx \\ &= C\varepsilon(r)^{(\alpha-\beta)q+n}\Big[1 - \big(r/\varepsilon(r)\big)^{(\alpha-\beta)q+n}\Big], \end{aligned}$$

which tends to infinity for r tending to zero. This shows that the norm in $L^q(v\,dx)$ of $I_\alpha f$ is equal to infinity.

On the other hand, from the definition of β we have $\beta p < 1$. Therefore,

$$\int |f|^p w\,dx \leqslant N\int_{|y|<1} |y|^{-\beta p}\,dy < \infty,$$

showing that f belongs to $L^p(w\,dx)$.

The estimates obtained show that (4.2) cannot hold. □

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